(a) Store the production function as $57K\text{^}(1 \div 4)*L\text{^}(3 \div 4)$. Store 277 as K and 743 as L. Evaluate to obtain $Q(277, 743)$; that is, the output when K is \$277,000 and L is 743 worker-hours. Repeat to obtain $Q(K, L)$ for the values of K and L in the following table:

***K*(\$1,000)**	**277**	**311**	**493**	**554**	**718**
L	**743**	**823**	**1,221**	**1,486**	**3,197**
***Q*(*K*, *L*)**					

(b) Note from the next to last column that the output $Q(277, 743)$ is doubled when K is doubled from 277 to 554 and L is doubled from 743 to 1,486. In a similar manner, verify that output is tripled when K and L are both tripled, and that output is halved when K and L are both halved. Does anything interesting happen if K is doubled and L is halved? Verify your response with your calculator.

48. Repeat Problem 47 with the production function

$$Q(K, L) = 83K^{2/5}L^{3/5}$$

2

Partial Derivatives

In many problems involving functions of two variables, the goal is to find the rate of change of the function with respect to one of its variables when the other is held constant. That is, the goal is to differentiate the function with respect to the particular variable in question while keeping the other variable fixed. This process is known as **partial differentiation,** and the resulting derivative is said to be a **partial derivative** of the function.

For example, suppose a manufacturer finds that

$$Q(x, y) = 5x^2 + 7xy$$

units of a certain commodity will be produced when x skilled workers and y unskilled workers are employed. Then if the number of unskilled workers remains fixed, the production rate with respect to the number of skilled workers is found by differentiating $Q(x, y)$ with respect to x while holding y constant. We call this **the partial derivative of Q with respect to x** and denote it by $Q_x(x, y)$; thus,

$$Q_x(x, y) = 5(2x) + 7(1)y = 10x + 7y$$

Similarly, if the number of skilled workers remains fixed, the production rate with respect to the number of unskilled workers is given by **the partial derivative of Q with respect to y,** which is obtained by differentiating $Q(x, y)$ with respect to y, holding x constant; that is, by

$$Q_y(x, y) = (0) + 7x(1) = 7x$$

Here is a general definition of partial derivatives and some alternative notation.

Partial Derivatives ■ Suppose $z = f(x, y)$. The partial derivative of f with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad f_x(x, y)$$

and is the function obtained by differentiating f with respect to x, treating y as a constant. The partial derivative of f with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad f_y(x, y)$$

and is the function obtained by differentiating f with respect to y, treating x as a constant.

Note

Recall from Chapter 2 that the derivative of a function of one variable $f(x)$ is defined by the limit of a difference quotient; namely,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

With this definition in mind, the partial derivative $f_x(x, y)$ is given by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and the partial derivative $f_y(x, y)$ by

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Computation of Partial Derivatives

No new rules are needed for the computation of partial derivatives. To compute f_x, simply differentiate f with respect to the single variable x, pretending that y is a constant. To compute f_y, differentiate f with respect to y, pretending that x is a constant. Here are some examples.

EXAMPLE 2.1

Find the partial derivatives f_x and f_y if $f(x, y) = x^2 + 2xy^2 + \dfrac{2y}{3x}$.

Solution

To simplify the computation, begin by rewriting the function as

$$f(x, y) = x^2 + 2xy^2 + \frac{2}{3}yx^{-1}$$

To compute f_x, think of f as a function of x and differentiate the sum term by term, treating y as a constant to get

$$f_x(x, y) = 2x + 2(1)y^2 + \frac{2}{3}y\,(-x^{-2}) = 2x + 2y^2 - \frac{2y}{3x^2}$$

To compute f_y, think of f as a function of y and differentiate term by term, treating x as a constant to get

$$f_y(x, y) = 0 + 2x(2y) + \frac{2}{3}(1)x^{-1} = 4xy + \frac{2}{3x}$$

EXAMPLE 2.2

Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = (x^2 + xy + y)^5$.

Solution

Holding y fixed and using the chain rule to differentiate z with respect to x, you get

$$\begin{aligned}\frac{\partial z}{\partial x} &= 5(x^2 + xy + y)^4 \frac{\partial}{\partial x}(x^2 + xy + y)\\ &= 5(x^2 + xy + y)^4(2x + y)\end{aligned}$$

Holding x fixed and using the chain rule to differentiate z with respect to y, you get

$$\begin{aligned}\frac{\partial z}{\partial x} &= 5(x^2 + xy + y)^4 \frac{\partial}{\partial y}(x^2 + xy + y)\\ &= 5(x^2 + xy + y)^4(x + 1)\end{aligned}$$

EXAMPLE 2.3

Find the partial derivatives f_x and f_y if $f(x, y) = xe^{-2xy}$.

Solution

From the product rule,

$$f_x(x, y) = x(-2ye^{-2xy}) + e^{-2xy} = (-2xy + 1)e^{-2xy}$$

and from the constant multiple rule,

$$f_y(x, y) = x(-2xe^{-2xy}) = -2x^2e^{-2xy}$$

Geometric Interpretation of Partial Derivatives

As you saw in Section 1, functions of two variables can be represented graphically as surfaces drawn on three-dimensional coordinate systems. In particular, if $z = f(x, y)$, an ordered pair (x, y) in the domain of f can be identified with a point in the xy plane and the corresponding function value $z = f(x, y)$ can be thought of as assigning a "height" to this point. The graph of f is the surface consisting of all points (x, y, z) in three-dimensional space whose height z is equal to $f(x, y)$.

The partial derivatives of a function of two variables can be interpreted geometrically as follows. For each fixed number y_0, the points (x, y_0, z) form a vertical plane whose equation is $y = y_0$. If $z = f(x, y)$ and if y is kept fixed at $y = y_0$, then the corresponding points $(x, y_0, f(x, y_0))$ form a curve in a three-dimensional space that is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$. At each point on this curve, the partial derivative $\frac{\partial z}{\partial x}$ is simply the slope of the line in the plane $y = y_0$ that is tangent to the curve at the point in question. That is, $\frac{\partial z}{\partial x}$ is the slope of the tangent "in the x direction." The situation is illustrated in Figure 7.10a.

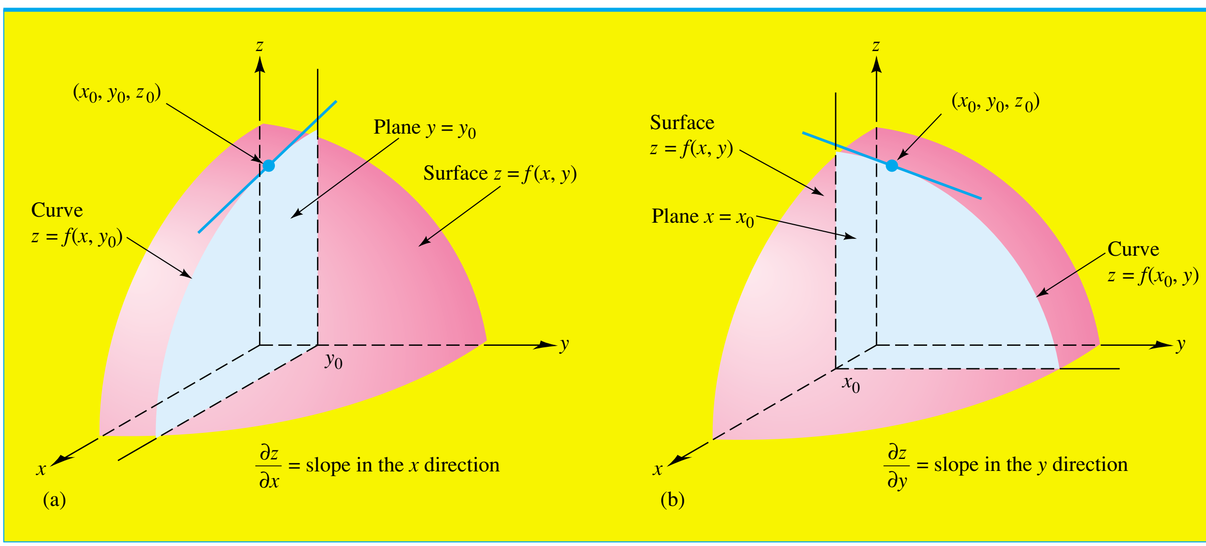

FIGURE 7.10 Geometric interpretation of partial derivatives.

Similarly, if x is kept fixed at $x = x_0$, the corresponding points $(x_0, y, f(x_0, y))$ form a curve that is the intersection of the surface $z = f(x, y)$ with the vertical plane $x = x_0$. At each point on this curve, the partial derivative $\frac{\partial z}{\partial y}$ is the slope of the tangent in the plane $x = x_0$. That is, $\frac{\partial z}{\partial y}$ is the slope of the tangent "in the y direction." The situation is illustrated in Figure 7.10b.

Marginal Analysis

In economics, the term marginal analysis refers to the practice of using a derivative to estimate the change in the value of a function resulting from a 1-unit increase in one of its variables. In Section 4 of Chapter 2, you saw some examples of marginal analysis involving ordinary derivatives of functions of one variable. Here is an example of how partial derivatives can be used in a similar fashion.

EXAMPLE 2.4

It is estimated that the weekly output of a certain plant is given by the function $Q(x, y) = 1{,}200x + 500y + x^2y - x^3 - y^2$ units, where x is the number of skilled workers and y the number of unskilled workers employed at the plant. Currently the workforce consists of 30 skilled workers and 60 unskilled workers. Use marginal analysis to estimate the change in the weekly output that will result from the addition of 1 more skilled worker if the number of unskilled workers is not changed.

Solution

The partial derivative

$$Q_x(x, y) = 1{,}200 + 2xy - 3x^2$$

is the rate of change of output with respect to the number of skilled workers. For any values of x and y, this is an approximation of the number of additional units that will be produced each week if the number of skilled workers is increased from x to $x + 1$ while the number of unskilled workers is kept fixed at y. In particular, if the workforce is increased from 30 skilled and 60 unskilled workers to 31 skilled and 60 unskilled workers, the resulting change in output is approximately

$$Q_x(30, 60) = 1{,}200 + 2(30)(60) - 3(30)^2 = 2{,}100 \text{ units}$$

For practice, compute the exact change $Q(31, 60) - Q(30, 60)$. Is the approximation a good one?

Recall that in Section 1 we introduced the *Cobb-Douglas production function*

$$Q(K, L) = AK^{\alpha}L^{\beta}$$

where Q is the output of a production process in which the capital investment is K thousand dollars and L worker-hours of labor are used. In this context, the partial derivative $Q_K(K, L)$ is called the **marginal productivity of capital** and measures the rate at which output Q changes with respect to capital expenditure when the size of the labor force is held constant. Similarly, $Q_L(K, L)$ is called the **marginal productivity of labor** and measures the rate of change of output with respect to the labor level when capital expenditure is held constant. The following example illustrates one way these partial derivatives can be used in economic analysis.

EXAMPLE 2.5

A manufacturer estimates that the monthly output at a certain factory is given by the Cobb-Douglas function

$$Q(K, L) = 50K^{0.4}L^{0.6}$$

where K is the capital expenditure in units of \$1,000 and L is the size of the labor force, measured in worker-hours.

(a) Find the marginal productivity of capital Q_K and the marginal productivity of labor Q_L when the capital expenditure is \$750,000, and the level of labor is 991 worker-hours.

(b) Should the manufacturer consider adding capital or increasing the labor level in order to increase output?

Solution

(a)

$$Q_K(K, L) = 50(0.4K^{-0.6})L^{0.6} = 20K^{-0.6}L^{0.6}$$

and

$$Q_L(K, L) = 50K^{0.4}(0.6L^{-0.4}) = 30K^{0.4}L^{-0.4}$$

so with $K = 750$ (\$750,000) and $L = 991$

$$Q_K(750, 991) = 20(750)^{-0.6}(991)^{0.6} \simeq 16.92$$

and

$$Q_L(750, 991) = 30(750)^{0.4}(991)^{-0.4} \simeq 33.54$$

(b) From part (a), you see that an increase in 1 unit of capital (that is, \$1,000) results in an increase in output of 16.92 units, which is less than the 33.54 unit increase in output that results from a unit increase in the labor level. Therefore, the manufacturer should increase the labor level by 1 worker-hour (from 991 worker-hours to 992) to increase output as quickly as possible from the current level.

Substitute and Complementary Commodities

Two commodities are said to be **substitute commodities** if an increase in the demand for either results in a decrease in demand for the other. Substitute commodities are competitive, like butter and margarine.

On the other hand, two commodities are said to be **complementary commodities** if a decrease in the demand of either results in a decrease in the demand of the other. An example is provided by cameras and film. If consumers buy fewer cameras, they likely buy less film, too.

We can use partial derivatives to obtain criteria for determining whether two commodities are substitute or complementary. Suppose $D_1(p_1, p_2)$ units of the first commodity and $D_2(p_1, p_2)$ of the second are demanded when the unit prices of the commodities are p_1 and p_2, respectively. It is reasonable to expect demand to decrease with increasing price, so

$$\frac{\partial D_1}{\partial p_1} < 0 \qquad \text{and} \qquad \frac{\partial D_2}{\partial p_2} < 0$$

For substitute commodities, the demand for each commodity increases with respect to the price of the other, so

$$\frac{\partial D_1}{\partial p_2} > 0 \qquad \text{and} \qquad \frac{\partial D_2}{\partial p_1} > 0$$

However, for complementary commodities, the demand for each decreases with respect to the price of the other, and

$$\frac{\partial D_1}{\partial p_2} < 0 \qquad \text{and} \qquad \frac{\partial D_2}{\partial p_1} < 0$$

Example 2.6 illustrates how these criteria can be used to determine whether a given pair of commodities are complementary, substitute, or neither.

EXAMPLE 2.6

Suppose the demand function for flour in a certain community is given by

$$D_1(p_1, p_2) = 500 + \frac{10}{p_1 + 2} - 5p_2$$

while the corresponding demand for bread is given by

$$D_2(p_1, p_2) = 400 - 2p_1 + \frac{7}{p_2 + 3}$$

where p_1 is the dollar price of a pound of flour and p_2 is the price of a loaf of bread. Determine whether flour and bread are substitute or complementary commodities or neither.

Solution

You find that

$$\frac{\partial D_1}{\partial p_2} = -5 < 0 \qquad \text{and} \qquad \frac{\partial D_2}{\partial p_1} = -2 < 0$$

Since both partial derivatives are negative for all p_1 and p_2, it follows that flour and bread are complementary commodities.

Second-Order Partial Derivatives

Partial derivatives can themselves be differentiated. The resulting functions are called **second-order partial derivatives.** Here is a summary of the definition and notation for the four possible second-order partial derivatives of a function of two variables.

Second-Order Partial Derivatives ■ If $z = f(x, y)$, the partial derivative of f_x with respect to x is

$$f_{xx} = (f_x)_x \qquad \text{or} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

The partial derivative of f_x with respect to y is

$$f_{xy} = (f_x)_y \qquad \text{or} \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

The partial derivative of f_y with respect to x is

$$f_{yx} = (f_y)_x \qquad \text{or} \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

The partial derivative of f_y with respect to y is

$$f_{yy} = (f_y)_y \qquad \text{or} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

The computation of second-order partial derivatives is illustrated in the next example.

EXAMPLE 2.7

Compute the four second-order partial derivatives of the function $f(x, y) = xy^3 + 5xy^2 + 2x + 1$.

Solution

Since

$$f_x = y^3 + 5y^2 + 2$$

it follows that

$$f_{xx} = 0 \qquad \text{and} \qquad f_{xy} = 3y^2 + 10y$$

Since

$$f_y = 3xy^2 + 10xy$$

it follows that

$$f_{yx} = 3y^2 + 10y \qquad \text{and} \qquad f_{yy} = 6xy + 10x$$

Note

The two partial derivatives f_{xy} and f_{yx} are sometimes called the **mixed second-order partial derivatives** of f. Notice that the mixed partial derivatives in Example 2.7 are equal. This is not an accident. It turns out that for virtually all functions $f(x, y)$ you will encounter in practical work, the mixed partials will be equal; that is,

$$f_{xy} = f_{yx}$$

This means you will get the same answer if you first differentiate $f(x, y)$ with respect to x and then differentiate the resulting function with respect to y as you would if you performed the differentiation in the reverse order.

In the next example, you will see how a second-order partial derivative can convey useful information in a practical situation.

EXAMPLE 2.8

Suppose the output Q at a factory depends on the amount K of capital invested in the plant and equipment and also on the size L of the labor force, measured in worker-hours. Give an economic interpretation of the sign of the second-order partial derivative $\dfrac{\partial^2 Q}{\partial L^2}$.

Solution

If $\dfrac{\partial^2 Q}{\partial L^2}$ is negative, the marginal product of labor $\dfrac{\partial Q}{\partial L}$ decreases as L increases. This implies that for a fixed level of capital investment, the effect on output of the addition of 1 worker-hour of labor is greater when the workforce is small than when the workforce is large.

Similarly, if $\dfrac{\partial^2 Q}{\partial L^2}$ is positive, it follows that for a fixed level of capital investment, the effect on output of the addition of 1 worker-hour of labor is greater when the workforce is large than when it is small.

For most factories operating with adequate workforces, the derivative $\frac{\partial^2 Q}{\partial L^2}$ is generally negative. Can you give an economic explanation for this fact?

PROBLEMS 7.2

In Problems 1 through 16, compute all first-order partial derivatives of the given function.

1. $f(x, y) = 2xy^5 + 3x^2y + x^2$

2. $z = 5x^2y + 2xy^3 + 3y^2$

3. $z = (3x + 2y)^5$

4. $f(x, y) = (x + xy + y)^3$

5. $f(s, t) = \frac{3t}{2s}$

6. $z = \frac{t^2}{s^3}$

7. $z = xe^{xy}$

8. $f(x, y) = xye^x$

9. $f(x, y) = \frac{e^{2-x}}{y^2}$

10. $f(x, y) = xe^{x+2y}$

11. $f(x, y) = \frac{2x + 3y}{y - x}$

12. $z = \frac{xy^2}{x^2y^3 + 1}$

13. $z = u \ln v$

14. $f(u, v) = u \ln uv$

15. $f(x, y) = \frac{\ln (x + 2y)}{y^2}$

16. $z = \ln \left(\frac{x}{y} + \frac{y}{x}\right)$

In Problems 17 through 20, evaluate the partial derivatives $f_x(x, y)$ *and* $f_y(x, y)$ *at the given point* $P_0(x_0, y_0)$.

17. $f(x, y) = 3x^2 - 7xy + 5y^3 - 3(x + y) - 1$; at $P_0(-2, 1)$

18. $f(x, y) = (x - 2y)^2 + (y - 3x)^2 + 5$; at $P_0(0, -1)$

19. $f(x, y) = xe^{-2y} + ye^{-x} + xy^2$; at $P_0(0, 0)$

20. $f(x, y) = xy \ln \left(\frac{y}{x}\right) + \ln (2x - 3y)^2$; at $P_0(1, 1)$

In Problems 21 through 26, find the second partials (including the mixed partials).

21. $f(x, y) = 5x^4y^3 + 2xy$

22. $f(x, y) = \frac{x + 1}{y - 1}$

23. $f(x, y) = e^{x^2y}$

24. $f(u, v) = \ln (u^2 + v^2)$

25. $f(s, t) = \sqrt{s^2 + t^2}$

26. $f(x, y) = x^2ye^x$

MARGINAL ANALYSIS **27.** At a certain factory, the daily output is $Q(K, L) = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours. Suppose that the current capital investment is \$900,000 and that 1,000 worker-hours of labor are used each day. Use marginal analysis to estimate the effect of an additional capital investment of \$1,000 on the daily output if the size of the labor force is not changed.

MARGINAL PRODUCTIVITY **28.** A manufacturer estimates that the annual output at a certain factory is given by

$$Q(K, L) = 30K^{0.3}L^{0.7}$$

units, where K is the capital expenditure in units of \$1,000 and L is the size of the labor force in worker-hours.

(a) Find the marginal productivity of capital Q_K and the marginal productivity of labor Q_L when the capital expenditure is \$630,000 and the labor level is 830 worker-hours.

(b) Should the manufacturer consider adding a unit of capital or a unit of labor in order to increase output more rapidly?

MARGINAL PRODUCTIVITY **29.** The productivity of a certain country is given by

$$Q(K, L) = 90K^{1/3}L^{2/3}$$

units, where K is the capital expenditure in units of \$1 million and L is the size of the labor force in thousands of worker-hours.

(a) Find the marginal productivity of capital Q_K and the marginal productivity of labor Q_L when the capital expenditure is \$5,495,000,000 ($K = 5{,}495$) and the labor level is 4,587,000 worker-hours ($L = 4{,}587$).

(b) Should the government of the country encourage capital investment or additional labor employment to increase productivity as rapidly as possible?

MARGINAL ANALYSIS **30.** A grocer's daily profit from the sale of two brands of apple juice is

$$P(x, y) = (x - 30)(70 - 5x + 4y) + (y - 40)(80 + 6x - 7y)$$

cents, where x is the price per can of the first brand and y is the price per can of the second. Currently the first brand sells for 50 cents per can and the second for 52 cents per can. Use marginal analysis to estimate the change in the daily profit that will result if the grocer raises the price of the second brand by 1 cent per can but keeps the price of the first brand unchanged.

FLOW OF BLOOD **31.** The smaller the resistance to flow in a blood vessel, the less energy is expended by the pumping heart. One of Poiseuille's laws* says that the resistance to the flow of blood in a blood vessel satisfies

* E. Batschelet, *Introduction to Mathematics for Life Scientists,* 2nd ed., Springer-Verlag, New York, 1976, page 279.

$$F(L, r) = \frac{kL}{r^4}$$

where L is the length of the vessel, r is its radius, and k is a constant that depends on the viscosity of blood.

(a) Find F, $\frac{\partial F}{\partial L}$ and $\frac{\partial F}{\partial r}$ in the case where $L = 3.17$ cm and $r = 0.085$ cm. Leave your answer in terms of k.

(b) Suppose the vessel in part (a) is constricted and lengthened so that its new radius is 20% smaller than before and its new length is 20% greater. How do these changes affect the flow $F(L, r)$? How do they affect the values of $\frac{\partial F}{\partial L}$ and $\frac{\partial F}{\partial r}$?

CONSUMER DEMAND

32. The monthly demand for a certain brand of toasters is given by a function $f(x, y)$, where x is the amount of money (measured in units of \$1,000) spent on advertising and y is the selling price (in dollars) of the toasters. Give economic interpretations of the partial derivatives f_x and f_y. Under normal economic conditions, what will be the sign of each of these derivatives?

CONSUMER DEMAND

33. A bicycle dealer has found that if 10-speed bicycles are sold for x dollars apiece and the price of gasoline is y cents per gallon, approximately $F(x, y)$ bicycles will be sold each month, where

$$F(x, y) = 200 - 24\sqrt{x} + 4(0.1y + 4)^{3/2}$$

Currently, the bicycles sell for \$324 apiece and gasoline sells for \$1.20. Use marginal analysis to determine the change in the demand for bicycles that results when the price of bicycles is kept fixed but the price of gasoline decreases by 1 cent per gallon.

SURFACE AREA OF THE HUMAN BODY

34. Recall from Problem 35 of Section 1 that the surface area of a person's body may be measured by the empirical formula

$$S(W, H) = 0.0072W^{0.425}H^{0.725}$$

where W (kg) and H (cm) are the person's weight and height, respectively. Currently a child weighs 50 kg and is 163 cm tall.

(a) Compute $S_W(50, 163)$ and $S_H(50, 163)$, and interpret each of these partial derivatives.

(b) Estimate the change in surface area that results if the child's height stays constant but her weight increases by 1 kg.

PACKAGING

35. A soft drink can is a cylinder H cm tall with radius R cm. Its volume is given by the formula $V = \pi R^2 H$. A particular can is 12 cm tall with radius 3 cm. Use calculus to estimate the change in volume that results if the radius is increased by 1 cm while the height remains at 12 cm.

PACKAGING

36. For the soft drink can in Problem 35, the surface area is given by $S = 2\pi R^2 + 2\pi RH$. Use calculus to estimate the change in surface that results if:
(a) The radius is increased from 3 cm to 4 cm while the height stays at 12 cm.
(b) The height is decreased from 12 to 11 cm while the radius stays at 3 cm.

SUBSTITUTE AND COMPLEMENTARY COMMODITIES

In Problems 37 through 42, the demand functions for a pair of commodities are given. Use partial derivatives to determine whether the commodities are substitute, complementary, or neither.

37. $D_1 = 500 - 6p_1 + 5p_2$; $D_2 = 200 + 2p_1 - 5p_2$

38. $D_1 = 1{,}000 - 0.02p_1^2 - 0.05p_2^2$; $D_2 = 800 - 0.001p_1^2 - p_1p_2$

39. $D_1 = 3{,}000 + \dfrac{400}{p_1 + 3} + 50p_2$; $D_2 = 2{,}000 - 100p_1 + \dfrac{500}{p_2 + 4}$

40. $D_1 = 2{,}000 + \dfrac{100}{p_1 + 2} - 25p_2$; $D_2 = 1{,}500 - \dfrac{p_2}{p_1 + 7}$

41. $D_1 = \dfrac{7p_2}{1 + p_1^2}$; $D_2 = \dfrac{p_1}{1 + p_2^2}$

42. $D_1 = 200p_1^{-1/2}p_2^{-1/2}$; $D_2 = 300p_1^{-1/2}p_2^{-3/2}$

LAPLACE'S EQUATION

The function z = f(x, y) *is said to satisfy* ***Laplace's equation*** *if*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Functions that satisfy such an equation play an important role in a variety of applications in the physical sciences, especially in the theory of electricity and magnetism. In Problems 43 through 46, determine whether the given function satisfies Laplace's equation.

43. $z = x^2 - y^2$

44. $z = xy$

45. $z = xe^y - ye^x$

46. $z = [(x - 1)^2 + (y + 3)^2]^{-1/2}$

CONSUMER DEMAND

47. Two competing brands of power lawnmowers are sold in the same town. The price of the first brand is x dollars per mower, and the price of the second brand is y dollars per mower. The local demand for the first brand of mower is given by a function $D(x, y)$.
(a) How would you expect the demand for the first brand of mower to be affected by an increase in x? By an increase in y?
(b) Translate your answers in part (a) into conditions on the signs of the partial derivatives of D.
(c) If $D(x, y) = a + bx + cy$, what can you say about the signs of the coefficients b and c if your conclusions in parts (a) and (b) are to hold?

CHEMISTRY **48.** The **ideal gas law** says that $PV = nRT$ for n moles of an ideal gas, where P is the pressure exerted by the gas, V is the volume of the gas, T is the temperature of the gas, and R is a constant (the **gas constant**). Compute the product

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}$$

CARDIOLOGY **49.** To estimate the amount of blood that flows through a patient's lung, cardiologists use the empirical formula

$$P(x, y, u, v) = \frac{100xy}{xy + uv}$$

where P is a percentage of the total blood flow, x is the carbon dioxide output of the lung, y is the arteriovenous carbon dioxide difference in the lung, u is the carbon dioxide output of the lung, and v is the arteriovenous carbon dioxide difference in the other lung.

It is known that blood flows into the lungs to pick up oxygen and dump carbon dioxide, so the arteriovenous carbon dioxide difference measures the extent to which this exchange is accomplished. (The actual measurement is accomplished by a device called a **cardiac shunt**.) The carbon dioxide is then exhaled from the lungs so that oxygen-bearing air can be inhaled.

Compute the partial derivatives P_x, P_y, P_u, and P_v, and give a physiological interpretation of each derivative.

MARGINAL PRODUCTIVITY **50.** Suppose the output Q of a factory depends on the amount K of capital investment measured in units of \$1,000 and on the size L of the labor force measured in worker-hours. Give an economic interpretation of the second-order partial derivative $\frac{\partial^2 Q}{\partial K^2}$.

MARGINAL PRODUCTIVITY **51.** At a certain factory, the output is $Q = 120K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours.

(a) Determine the sign of the second-order partial derivative $\frac{\partial^2 Q}{\partial L^2}$ and give an economic interpretation.

(b) Determine the sign of the second-order partial derivative $\frac{\partial^2 Q}{\partial K^2}$ and give an economic interpretation.

LAW OF DIMINISHING RETURNS **52.** Suppose the daily output Q of a factory depends on the amount K of capital investment and on the size L of the labor force. A **law of diminishing returns** states that in certain circumstances, there is a value L_0 such that the marginal product of labor will be increasing for $L < L_0$ and decreasing for $L > L_0$.

(a) Translate this law of diminishing returns into statements about the sign of a certain second-order partial derivative.

(b) Read about the principle of diminishing returns in an economics text. Then write a paragraph discussing the economic factors that might account for this phenomenon.

53. It is estimated that the weekly output at a certain plant is given by $Q(x, y) = 1{,}175x + 483y + 3.1x^2y - 1.2x^3 - 2.7y^2$ units, where x is the number of skilled workers and y is the number of unskilled workers employed at the plant. Currently the workforce consists of 37 skilled and 71 unskilled workers.

(a) Store the output function as

1,175X + 483Y + 3.1(X^2)*Y − 1.2(X^3) − 2.7(Y^2)

Store 37 as X and 71 as Y and evaluate to obtain $Q(37, 71)$. Repeat for $Q(38, 71)$ and $Q(37, 72)$.

(b) Store the partial derivative $Q_x(x, y)$ in your calculator and evaluate $Q_x(37, 71)$. Use the result to estimate the change in output resulting when the workforce is increased from 37 skilled workers to 38 and the unskilled workforce stays fixed at 71. Then compare with the actual change in output, given by the difference $Q(38, 71) - Q(37, 71)$.

(c) Use the partial derivative $Q_y(x, y)$ to estimate the change in output that results when the number of unskilled workers is increased from 71 to 72 while the number of skilled workers stays at 37. Compare with $Q(37, 72) - Q(37, 71)$.

54. Repeat Problem 53 with the output function

$$Q(x, y) = 1{,}731x + 925y + x^2y - 2.7x^2 - 1.3y^{3/2}$$

and initial employment levels of $x = 43$ and $y = 85$.

3 Optimizing Functions of Two Variables

Suppose a manufacturer produces two VCR models, the deluxe and the standard, and that the total cost of producing x units of the deluxe and y units of the standard is given by the function $C(x, y)$. How would you find the level of production $x = a$ and $y = b$ that results in minimal cost? Or perhaps the output of a certain production process is given by $Q(K, L)$, where K and L measure capital and labor expenditure, respectively. What levels of expenditure K_0 and L_0 result in maximum output?

In Section 4 of Chapter 3, you learned how to use the derivative $f'(x)$ to find the largest and smallest values of a function of a single variable $f(x)$, and the goal of this section is to extend those methods to functions of two variables $f(x, y)$. We begin with a definition.